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# GULYAEV-BLEUSTEIN WAVES IN PIEZOELECTRIC MEDIA*

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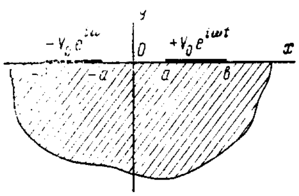
A solution is given for the problem of the excitation of Gulyaev-Bleustein shear surface acoustic waves by two ribbon electrodes of finite length on the surface of a semi-infinite crystal of hexagonal class 6mm. The electron charge distribution density functions are determined on the electrodes as are also the shear surface-wave characteristics.

The formulation of the selfconsistent problem of the excitation of surface waves in a piezoelectric medium by a system of metal electrodes is elucidated in general form in /1/, and the method of solving it is based on using Green's matrix for the linear charge on the piezoelectric surface. By using Green's matrix, Fredholm singular integral equations of the first kind were obtained in /2, 3/ for the unknown electric-charge distribution density functions on the electrodes. The integral Eqs./2, 3/ allow of an analytic solution when investigating the excitation of shear waves in a hexagonal crystal by a system of narrow electrodes.

1. We consider an elastic semi-infinite crystal of the hexagonal class 6mm that occupies the domain $y < 0$, $|x| < \infty$ ($z$ is the hexagonal axis) (figure). Two metal electrodes of identical width and infinite length in the direction of the $z$ axis are deposited on the boundary between the half-space and the vacuum ($y = 0$), and an alternating voltage $\pm V_0 e^{i\omega t}$ is applied thereto. In this case pure shear electroelastic waves with just one displacement component $w \neq 0$ ($w = w(x, y)$, $u = 0$, $v = 0$) exist within the half-space $y < 0$.




The equations of state for a 6mm crystal have the form /4/

$$\tau_{xz} = c_{44}^E \frac{\partial w}{\partial x} + e_{15}\frac{\partial \varphi}{\partial x}, \quad \tau_{yz} = c_{44}^E \frac{\partial w}{\partial y} + e_{15}\frac{\partial \varphi}{\partial y} \tag{1.1}$$
$$D_x = e_{15}\frac{\partial w}{\partial x} - \varepsilon_{11}^S\frac{\partial \varphi}{\partial x}, \quad D_y = e_{15}\frac{\partial w}{\partial y} - \varepsilon_{11}^S\frac{\partial \varphi}{\partial y}$$
$$(E_x = -\partial\varphi/\partial x,\ E_y = -\partial\varphi/\partial y)$$

Here $c_{44}^E$ is the electric modulus, $e_{15}$ is the piezoelectric constant, $\varepsilon_{11}^S$ is the permittivity, and $\varphi$ is the electric field governing the components of the electric field vector for $y < 0$.

Taking (1.1), into account, we obtain the fundamental equations for the amplitude components of the shear acoustoelectric waves (the time factor $e^{i\omega t}$ is omitted) from the equations of motion and electrostatics, where $\varkappa^2$ is the electromechanical coupling coefficient

$$\nabla^2 w + k^2 w = 0,\ \nabla^2\Phi = 0 \tag{1.2}$$
$$\left(k^2 = \frac{\rho\omega^2}{c_{44}^E(1+\varkappa^2)},\quad \varkappa^2 = \frac{e_{15}^2}{\varepsilon_{11}^S c_{44}^E},\quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right.$$
$$\left.\varphi = \frac{e_{15}}{\varepsilon_{11}^S} w + \Phi\right)$$

We have for the vacuum region with permittivity $\varepsilon_0$

$$\nabla^2\varphi_0 = 0 \tag{1.3}$$

where $\varphi_0$ is the electric potential for $y > 0$.

Taking account of the symmetry of the electroelastic state, we can represent the solution of (1.2), (1.3) in the form

$$w(x, y) = \langle a\rangle,\ \Phi(x, y) = \langle b\rangle,\ y < 0 \tag{1.4}$$
$$\varphi_0(x, y) = \langle b_0\rangle,\ y > 0 \tag{1.5}$$

Here

$$a = a(\xi, x, y) = A(\xi)\, e^{y\sqrt{\xi^2-k^2}} \sin \xi x, \quad b = b(\xi, x, y) = B(\xi)\, e^{\xi y} \sin \xi x$$

$$b_0 = b_0(\xi, x, y) = B_0(\xi)\, e^{-\xi y} \sin \xi x, \quad \langle c\rangle = \int_0^\infty c(\xi, x, y)\, d\xi,$$

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$$\sqrt{\xi^2-k^2}=\begin{cases}\sqrt{\xi^2-k^2}, & \xi>k\\ i\sqrt{k^2-\xi^2}, & \xi<k\end{cases}$$

We then obtain for the stress $\tau_{yz}$, the electric potential $\varphi$, and the component $D_y$ of the electric induction vector for $y<0$, as well as for the normal component $D_y^{(0)}$ of the induction vector for $y>0$

$$\tau_{yz}(x,y)=c_{44}^E(1+\varkappa^2)\langle\sqrt{\xi^2-k^2}a\rangle+e_{15}\langle\xi b\rangle \tag{1.6}$$

$$\varphi(x,y)=\frac{e_{15}}{\varepsilon_{11}^S}\langle a\rangle+\langle b\rangle$$

$$D_y(x,y)=-\varepsilon_{11}^S\langle\xi b\rangle,\quad D_y^{(0)}(x,y)=\varepsilon_0\langle\xi b_0\rangle \tag{1.7}$$

The following boundary conditions must be satisfied on the free surface of the crystal $y=0$:

$$\tau_{yz}(x,0)=0,\quad \varphi(x,0)=\varphi_0(x,0),\quad 0\leqslant x<\infty \tag{1.8}$$

$$\varphi(x,0)=\varphi_0(x,0)=V_0,\quad a<x<b \tag{1.9}$$

$$D_y(x,0)-D_y^{(0)}(x,0)=0,\quad 0\leqslant x<a,\quad x>b$$

Substituting the relationships (1.5) and (1.6) into condition (1.8) we obtain

$$A(\xi)=-\frac{e_{15}}{c_{44}^E(1+\varkappa^2)}\frac{\xi B(\xi)}{\sqrt{\xi^2-k^2}} \tag{1.10}$$

$$B_0(\xi)=B(\xi)\left[1-\frac{\bar{\varepsilon}\varkappa^2\xi}{(1+\varkappa^2)\sqrt{\xi^2-k^2}}\right]$$

Taking these into account, we obtain three integral equations in the function $B_*(\xi)=B(\xi)F(\xi)$ from conditions (1.9)

$$\int_0^\infty B_*(\xi)\left[1-\frac{\bar{\varepsilon}_{11}\delta\xi}{\sqrt{\xi^2-k^2}F(\xi)}\right]\sin\xi x\,d\xi=V_0,\quad a<x<b \tag{1.11}$$

$$-\varepsilon_0(1+\bar{\varepsilon}_{11})\int_0^\infty \xi B_*(\xi)\sin\xi x\,d\xi=0,\quad 0\leqslant x<a,\quad x>b$$

Here

$$F(\xi)=1-\frac{\delta\xi}{\sqrt{\xi^2-k^2}},\quad \delta=\frac{\varkappa^2}{(\bar{\varepsilon}_{11}+1)(1+\varkappa^2)},\quad \bar{\varepsilon}_{11}=\frac{\varepsilon_{11}^S}{\varepsilon_0}$$

The problem therefore reduces to determining the function $B_*(\xi)$ from the solution of the three integral Eq. (1.11).

We note that the left side of the second equation in (1.11) for $a<x<b$ is the unknown electric charge density distribution function $q(x)$ on the electrode, and, therefore by inverting the equation

$$-\varepsilon_0(1+\bar{\varepsilon}_{11})\int_0^\infty \xi B_*(\xi)\sin\xi x\,d\xi=\begin{cases}q(x), & a<x<b,\\ 0, & 0\leqslant x<a,\quad x>b\end{cases} \tag{1.12}$$

we obtain

$$\xi B_*(\xi)=-\frac{Q(\xi)}{\varepsilon_0(1+\bar{\varepsilon}_{11})},\quad Q(\xi)=\frac{2}{\pi}\int_a^b q(\eta)\sin\xi\eta\,d\eta \tag{1.13}$$

Now using relationships (1.10) and (1.13), we can write the formula for the displacement and the electric potential in the piezoelectric domain. The integrands in these formulas have a simple pole $\xi=\xi_0=k(1-\delta^2)^{-1/2}$ on the axis of integration since $F(\xi_0)=0$ for $\xi_0=k(1-\delta)^{-1/2}$. The existence of this pole is due to the fact that acoustoelectric wave damping is not taken into account in this problem. If wave damping is taken into account, we should set $k=k'-ik''$ $(k''>0)$ and then $\xi_0=\xi_0'-i\xi_0''$ $(\xi_0''>0)$. Therefore, by extracting the contribution from the pole $\xi=\xi_0$ the path of integration must be selected so that this pole is bypassed from above. Performing the necessary calculations, we obtain (the integrals are understood in the principal-value sense)

$$w(x,y)=\frac{\delta\bar{\varepsilon}_{11}}{e_{15}}\int_0^\infty\frac{Q(\xi)e^{y\sqrt{\xi^2-k^2}}}{F(\xi)\sqrt{\xi^2-k^2}}\sin\xi x\,d\xi-\frac{\pi\delta\bar{\varepsilon}_{11}Q(\xi_0)e^{y\sqrt{\xi_0^2-k^2}}}{e_{15}F'(\xi_0)\sqrt{\xi_0^2-k^2}}\,i\sin\xi_0 x \tag{1.14}$$

$$\varphi(x,y)=-\frac{1}{\varepsilon_0(1+\bar{\varepsilon}_{11})}\int_0^\infty\left[e^{\xi y}-\frac{\varkappa^2\xi e^{y\sqrt{\xi^2-k^2}}}{(1+\varkappa^2)\sqrt{\xi^2-k^2}}\right]\frac{Q(\xi)\sin\xi x}{F(\xi)\,\xi}\,d\xi+$$

$$\frac{\pi}{\varepsilon_0(1+\bar{\varepsilon}_{11})}\left[e^{\xi_0 y}-(1+\bar{\varepsilon}_{11})e^{y\sqrt{\xi_0^2-k^2}}\right]\frac{Q(\xi_0)}{F'(\xi_0)\,\xi_0}\,i\sin\xi_0 x$$

The last components in (1.14) correspond to Gulyaev-Bleustein surface waves are propagate in the positive and negative directions of the $x$-axis. In particular, for a surface wave propagating in the positive $x$-direction, expressions are obtained from (1.14) for the displacement and electric potential that differ by the factor $-{}^1/_2 \exp[i(\omega t - \xi_0 x)]$ instead of $i \sin \xi_0 x$ from the last terms in (1.14). On the basis of these expressions, we can write formulas for the stress and electric induction vector components in the surface wave

$$\tau_{xz}^{(w)} = -\frac{\pi i e_{15}}{2\varepsilon_0(1+\bar{\varepsilon}_{11})}\left[(1+\bar{\varepsilon}_{11})\frac{(1+\varkappa^2)}{\varkappa^2}e^{y\sqrt{\xi_0^2-k^2}} - e^{\xi_0 y}\right]\frac{Q(\xi_0)}{F'(\xi_0)}e^{i(\omega t-\xi_0 x)} \tag{1.15}$$

$$\tau_{yz}^{(w)} = \frac{\pi e_{15}}{2\varepsilon_0(1+\bar{\varepsilon}_{11})}\left[e^{y\sqrt{\xi_0^2-k^2}} - e^{\xi_0 y}\right]\frac{Q(\xi_0)}{F'(\xi_0)}e^{i(\omega t-\xi_0 x)}$$

$$D_x^{(w)} = -iD_y^{(w)} = -\frac{\pi i \bar{\varepsilon}_{11}}{2(1+\bar{\varepsilon}_{11})}\frac{Q(\xi_0)e^{\xi_0 y}}{F'(\xi_0)}e^{i(\omega t-\xi_0 x)}$$

Expressions (1.14) and (1.15) now enable us to determine the flux density vector of the energy transportable by the Gulyaev-Bleustein wave from the electrode emitter. The total energy flux density is the sum of the flux densities of the elastic energy $P_k{}^s$ and the electromagnetic energy $P_k{}^E$ (in the quasistatic approximation /1/)

$$P_k = P_k{}^s + P_k{}^E, \quad P_k{}^s = -\tau_{ik}\frac{\partial u_i}{\partial t}, \quad P_k{}^E = \varphi\frac{\partial D_k}{\partial t}$$

For monochromatic plane waves (with a time factor $e^{i\omega t}$), the quantities $P_k{}^s$ and $P_k{}^E$ can be averaged over the period of the vibrations. Then we find the components $\bar{P}_x$, $\bar{P}_y$ of the energy flux density vector for a shear surface wave (the asterisk denotes the complex conjugate quantity)

$$\bar{P}_x = \frac{\omega}{2}\operatorname{Im}[\tau_{xz}^* w - \varphi^* D_x], \quad \bar{P}_y = \frac{\omega}{2}\operatorname{Im}[\tau_{yz}^* w - \varphi^* D_y] \tag{1.16}$$

Substituting the amplitude values of the corresponding quantities from (1.14) and (1.15) into (1.16), we obtain

$$\bar{P}_x = \frac{\pi\omega|Q(\xi_0)|^2\bar{\varepsilon}_{11}}{8\varepsilon_0[F'(\xi_0)]^2(1+\bar{\varepsilon}_{11})^2}\left[\frac{(\bar{\varepsilon}_{11}+1)}{\sqrt{\xi_0^2-k^2}}e^{2y\sqrt{\xi_0^2-k^2}} - \frac{e^{2\xi_0 y}}{\xi_0}\right], \quad \bar{P}_y = 0 \tag{1.17}$$

**2.** We now turn to the solution of the three integral equations (1.11), which enables us to determine the electric-charge distribution density function on the electrode $q(x)$ and the quantity $Q(\xi_0)$ in (1.14), (1.15), and (1.17). We represent the function $q(x)$ in the form of a series ($T_n(z)$ are Chebyshev polynomials of the first kind)

$$q(x) = \frac{2\varepsilon_0(1+\bar{\varepsilon}_{11})V_0}{\sqrt{(b-x)(x-a)}}\sum_{n=0}^{\infty}a_n T_n\left(2\frac{x-a}{b-a}-1\right), \quad a<x<b \tag{2.1}$$

We find from (1.12)

$$\xi B_*(\xi) = -\frac{4V_0}{\pi}\sum_{n=0}^{\infty}a_n\int_a^b T_n\left(2\frac{x-a}{b-a}-1\right)\sin\xi x\frac{d\xi}{\sqrt{(b-x)(x-a)}} \tag{2.2}$$

Introducing a change of variable of intergration in (2.2) and taking account of the relationship between Chebyshev polynomials and Bessel functions /5/, we obtain

$$\xi B_*(\xi) = -2V_0\sum_{n=0}^{\infty}a_n S_n\left(\xi\frac{b+a}{2}\right)J_n\left(\xi\frac{b-a}{2}\right) \tag{2.3}$$

$$S_n\left(\xi\frac{b+a}{2}\right) = \left\{[1-(-1)^n](-1)^{(n-1)/2}\cos\left(\xi\frac{b+a}{2}\right) + [1+(-1)^n](-1)^{n/2}\sin\left(\xi\frac{b+a}{2}\right)\right\}$$

Now substituting the first equation of (2.3) into (1.11), we make the change of variable

$$x = \frac{b-a}{2}x_1 + \frac{b+a}{2}\,(|x_1|<1)$$

and we use the expansion

$$\sin\xi x = \sum_{m=0}^{\infty}\varepsilon_m J_m\left(\xi\frac{b-a}{2}\right)S_m\left(\xi\frac{b+a}{2}\right)T_m(x_1)$$

$$\varepsilon_0 = {}^1/_2, \quad \varepsilon_m = 1 \quad (m=1,2,\ldots)$$

We consequently obtain an infinite system of algebraic equations to determine the coefficients $a_n$ of the expansion (2.1)

$$\sum_{n=0}^{\infty} a_n(\omega_{nm} - \bar{\varepsilon}_{11}\delta\gamma_{nm}) = -\delta_{m0} \quad (m = 0, 1, 2\ldots) \tag{2.4}$$

Here

$$\omega_{nm} = \int_0^\infty J_n(\eta) J_m(\eta) S_n\left(\frac{\eta}{\alpha}\right) S_m\left(\frac{\eta}{\alpha}\right)\frac{d\eta}{\eta} \tag{2.5}$$

$$\gamma_{nm} = \int_0^\infty \frac{J_n(\eta) J_m(\eta)}{\sqrt{\eta^2 - k_1^2}F_1(\eta)} S_n\left(\frac{\eta}{\alpha}\right) S_m\left(\frac{\eta}{\alpha}\right) d\eta$$

$$\alpha = \frac{b-a}{b+a}, \quad k_1^2 = k^2\frac{b-a}{2}, \quad F_1(\eta) = 1 - \frac{\delta\eta}{\sqrt{\eta^2 - k_1^2}}$$

A new variable of integration $\eta = \xi(b-a)/2$ is introduced to transform the integrals $\omega_{nm}, \gamma_{nm}$

Since

$$S_{2n}\left(\frac{\eta}{\alpha}\right) = 2(-1)^n \sin\frac{\eta}{\alpha}$$

$$S_{2n-1}\left(\frac{\eta}{\alpha}\right) = 2(-1)^n \cos\frac{\eta}{\alpha} \qquad (n = 0, 1, 2, \ldots)$$

then the values of the following integrals

$$I_{2n,\,2m} = \int_0^\infty J_{2n}(\eta) J_{2m}(\eta) \sin^2\frac{\eta}{\alpha}\frac{d\eta}{\eta} \tag{2.6}$$

$$I_{2n-1,\,2m-1} = \int_0^\infty J_{2n-1}(\eta) J_{2m-1}(\eta) \cos^2\frac{\eta}{\alpha}\frac{d\eta}{\eta}$$

$$I_{2n+1,\,2m} = \int_0^\infty J_{2n+1}(\eta) J_{2m}(\eta) \sin\frac{\eta}{\alpha}\cos\frac{\eta}{\alpha}\frac{d\eta}{\eta} \quad (n, m = 0, 1, 2, \ldots)$$

must be found to evaluate the coefficients of $\omega_{nm}$

The integrals (2.6) can be converted to a form convenient for calculations if the Neumann formula /5/ is used

$$J_n(\eta) J_m(\eta) = \frac{2}{\pi}\int_0^{\pi/2} J_{n+m}(2\eta\cos\theta)\cos(n-m)\theta\, d\theta$$

as well as values of the discontinuous integrals

$$\int_0^\infty J_\mu(a\eta)\begin{Bmatrix}\cos\\ \sin\end{Bmatrix} b\eta\,\frac{d\eta}{\eta} = \frac{a^\mu}{\mu}\begin{Bmatrix}\cos\\ \sin\end{Bmatrix}\frac{\pi\mu}{2}(b + \sqrt{b^2 - a^2})^{-\mu} \quad (b > a)$$

In particular, for $n + m > 0$

$$I_{2n,\,2m} = \frac{2}{\pi}\int_0^{\pi/2}\cos(2n-2m)\theta\int_0^\infty J_{2n+2m}(2\eta\cos\theta)\sin^2\frac{\eta}{\alpha}\frac{d\eta}{\eta}\,d\theta =$$

$$\frac{(-1)^{n+m+1}\alpha^{2n+2m}}{\pi(2n+2m)}\int_0^{\pi/2}\cos[(2n-2m)\theta]\left(\frac{\cos\theta}{\beta(\theta)}\right)^{2n+2m} d\theta +$$

$$\begin{cases} 1/(8n), & n = m\\ 0, & n \neq m\end{cases}$$

$$(\beta(\theta) = 1 + \sqrt{1 - \alpha^2\cos^2\theta})$$

Transforming the other integrals of (2.6) in an analogous manner, we obtain

$$\omega_{nm} = \frac{\delta_{nm}}{n} - \frac{(-1)^{n+m}\alpha^{n+m}}{(n+m)}\frac{4}{\pi}\int_0^{\pi/2}\cos(n-m)\theta\,\times \tag{2.7}$$

$$\left(\frac{\cos\theta}{\beta(\theta)}\right)^{n+m} d\theta \quad (n + m > 0)$$

$$\omega_{00} = \frac{4}{\pi}\int_0^{\pi/2}\ln\beta(\theta)\,d\theta + 2\ln\frac{2}{\alpha} \tag{2.8}$$

We note that for narrow electrodes ($\alpha \ll 1$) it follows from (2.7) and (2.8) that

$$\omega_{00} \approx 2\ln(4/\alpha),\quad \omega_{nm} \approx \delta_{nm}/n\quad (n+m>0)$$

In evaluating the integrals (2.7) it must be taken into account that the integrand has a simple pole at the point $\eta_0 = k_1/\sqrt{1-\delta^2}$. Bypassing this pole from above, we obtain (the second integral is understood in the principal-value sense)

$$\gamma_{nm} = \int_0^{k_1} \frac{J_n(\eta)\,J_m(\eta)}{i\sqrt{k_1^2-\eta^2}-\delta\eta} S_n\left(\frac{\eta}{\alpha}\right) S_m\left(\frac{\eta}{\alpha}\right) d\eta + \int_{k_1}^{\infty} \frac{J_n(\eta)\,J_m(\eta)}{\sqrt{\eta^2-k_1^2}-\delta\eta} S_n\left(\frac{\eta}{\alpha}\right) S_m\left(\frac{\eta}{\alpha}\right) d\eta - \frac{i\pi\delta}{1-\delta^2} J_n(\eta_0)\,J_m(\eta_0)\,S_n\left(\frac{\eta_0}{\alpha}\right) S_m\left(\frac{\eta_0}{\alpha}\right) \tag{2.9}$$

For piezoelectric materials of the hexagonal class 6mm the permittivity $\varepsilon_{11}^S$ is considerably greater than $\varepsilon_0$; consequently, the parameter $\delta$ is small, and therefore, (2.10) can be used to evaluate the coefficients $\gamma_{nm}$

$$\begin{aligned} \gamma_{nm} &= \gamma_{nm}' - i\gamma_{nm}'' \\ \gamma_{nm}' &= \int_0^{\infty} J_n(k_1 \operatorname{ch}\xi)\,J_m(k_1 \operatorname{ch}\xi)\,S_n\left(\frac{k_1}{\alpha}\operatorname{ch}\xi\right) S_m\left(\frac{k_1}{\alpha}\operatorname{ch}\xi\right) d\xi \\ \gamma_{nm}'' &= \int_0^{\pi/2} J_n(k_1\sin\varphi)\,J_m(k_1\sin\varphi)\,S_n\left(\frac{k_1}{\alpha}\sin\varphi\right) S_m\left(\frac{k_1}{\alpha}\sin\varphi\right) d\varphi \end{aligned} \tag{2.10}$$

(the change in the variable of integration $\eta = k_1 \operatorname{ch}\xi$, $\eta = k_1\sin\varphi$) was made).

Therefore, the solution of the problem is reduced to an infinite system of linear algebraic equations (2.4). The fundamental parameter $Q(\xi_0)$, governing the Gulyaev-Bleustein surface-wave characteristic, is here expressed in terms of the solution of this system by means of the formula

$$Q(\xi_0) = 2\varepsilon_0(1+\bar{\varepsilon}_{11})V_0 \sum_{n=0}^{\infty} a_n J_n\left(\frac{k_1}{\sqrt{1-\delta^2}}\right) S_n\left(\frac{k_1}{\alpha\sqrt{1-\delta^2}}\right) \tag{2.11}$$

An an illustration, an electronic computer claculation of the quantity $\bar{Q} = 10^3 Q(\xi_0)/(2\varepsilon_0(1+\bar{\varepsilon}_{11})V_0)$ was performed for the piezoelectric material CdS with the characteristics /4/ $e_{15} = -0.21$ C/m$^2$, $\varepsilon_{11}^S = 8\cdot10^{-11}$ F/m, $c_{44}^E = 1.49\cdot10^{10}$ N/m$^2$, $\varkappa^2 = 0.037$, $\delta = 3.57\cdot10^{-3}$.

Values of $\bar{Q}$ calculated for certain values of $k_1$ and $\alpha$ are represented below, after truncating the infinite system of algebraic Eqs.(2.4) and replacing it by a system of four equations

| | $\alpha$ | 0.2 | 0.4 | 0.6 | 0.8 |
|---|---|---|---|---|---|
| $\bar{Q}$ | $(k_1=2)$ | $23.2-0.729i$ | $1(6-4.16i$ | $-89.5+5.67i$ | $-256+10.8i$ |
| $\bar{Q}$ | $(k_1=4)$ | $123-2.13i$ | $-94.3+1.25i$ | $75.9+0.109i$ | $97.4-1.47i$ |

A numerical analysis on a computer showed that when the number of equations increased above four, the refinement in the values of $\bar{Q}$ was negligible.

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